Partial Differential Equations - Final exam

You have 3 hours to complete this exam. Please show all work. The exam consists of 4 questions for a total of 90 points. You get 10 additional points for putting your name and student ID on every page you wish to be graded bringing the total to 100 points. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

(1) (24 points) The initial boundary problem

$$u(t,0) = u_{xx}(t,0) = u(t,1) = u_{xx}(t,1) = 0 \quad 0 < x < 1$$
$$u_{tt} = -u_{xxxx} \quad u(0,x) = f(x) \quad u_t(0,x) = 0 \quad t > 0$$

models the vibrations of an elastic beam of unit length with simply supported ends, subject to a nonzero initial displacement $f(x)$ and zero initial velocity.

(a) (12 points) What are the vibrational frequencies for the beam?

(b) (12 points) Write down the solution to the initial boundary value problem as a Fourier series.

(2) (24 points) Consider the following problem

$$\partial_{tt} u = c^2 \partial_{xx} u, \quad t > 0$$
$$u(0,x) = e^{-x^2},$$
$$\partial_t u(0,x) = f(x).$$

for c a constant.

(a) (12 points) Let $x \in (0,1)$ and $f(x) = \sin(x)$. Find u using D'Alembert's principle.

(b) (12 points) Let $x \in \mathbb{R}$ and $f(x) = 0$. Find u using Fourier transform. (Hint: remember the duality principle and the shift theorem).

(3) (20 points)

(a) (10 points) State the the harmonic maximum principle for bounded simply connected domains Ω with smooth boundary.

(b) (10 points) Prove that solutions $u(x)$ to Poisson's equation augmented with boundary conditions:

$$\Delta u = f \quad u|_{\partial\Omega} = h.$$

with $f(x)$ in $C^2(\Omega)$, $h(x)$ in $C^2(\partial\Omega)$ are unique.

(4) (22 points) Consider the linear transport equation $u_t + (1 + x^2)u_x = 0$

(a) (8 points) Find the characteristic curves.

(b) (7 points) Write down a formula for the general solution.

(c) (7 points) Find the solution to the initial value problem $u(0,x) = f(x)$ and discuss the solution behavior as t increases.